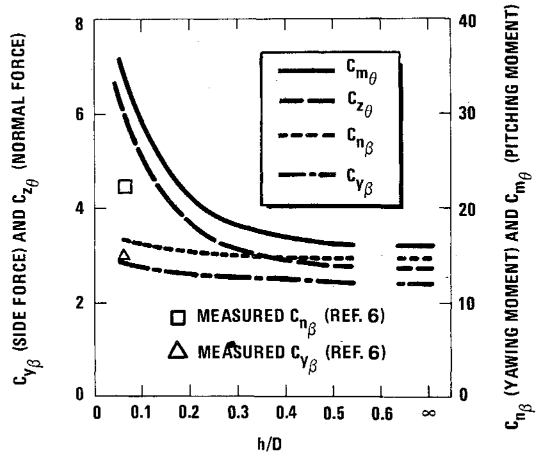

Fig. 3 Displacement derivatives of lateral and vertical force and moment coefficients.

found, a_y and a_z are readily integrated analytically, providing all cross sections are similar, to obtain the vehicle force and moment coefficients.

Method of Solution

Since potential-flow solutions for bodies near a plane, even for elementary geometries, are not readily obtainable in closed form, a modification of the numerical method of Hess and Smith[5] was used to compute ϕ_y and ϕ_z. The solution is cast in the form of a distribution over the surface of source singularities of unknown strength, together with an image distribution. Thus, ϕ_y, for example, is given by

$$\phi_y(y, z;x) = \frac{1}{2\pi}\int_s \sigma_y(s)\ \ln(r\bar{r})ds$$

where r is the distance from a point (η,ζ) on the surface to the field point (y,z) and $\bar{r}$ is the distance from the corresponding point $(\eta,\zeta\ -2H)$ on the image surface to (y,z). To approximate the effect of the continuous distribution, the two surfaces are replaced by arrays of N rectilinear elements. The source strength is taken to be constant on each element. Imposition of the flow-tangency condition at the midpoint of each element results in N linear algebraic equations which are solved to obtain the N source strengths.

Results of Computations-Comparison with Experiment

Computations of the derivatives of both lateral and longitudinal force and moment coefficients were carried out for the vehicle geometry sketched in Fig. 2. This configuration was selected because results of measurements of static lateral force and moment coefficients, for this vehicle, near a ground plane, are available from Ref. 6. In the calculations, the base area was assumed to be the maximum cross-sectional area of the vehicle, because it is indicated in Ref. 6 that the flow was separated over the tail during the tests.

The computed variation with gap height of the derivatives of the force and moment coefficients with respect to angular displacement are plotted in Fig. 3. Derivatives with respect to dimensionless angular rates $\theta' = \dot{\theta}D/V$ and $\beta' = \dot{\beta}D/V$ are shown in Fig. 2. Reference area and length are, respectively, maximum cross-sectional area and width of the vehicle.

Ground effect is seen to greatly amplify the longitudinal loading, as would be expected for this configuration. It has a smaller but still significant effect on the lateral loading as well. The aforementioned experimental results for the lateral loading coefficients, obtained for a gap-to-vehicle-width ratio of 0.05, are indicated by the two data points on Fig. 3. The theoretical and experimental moment derivatives are seen to differ by about 30% and the force derivatives by only 3%. While the close agreement of the lateral force coefficients is, perhaps, forfuitous, in light of the uncertainty in the value of effective base area, it can be concluded that the method provides reasonably accurate and rapid estimates of the loading due to response, including the effects of ground proximity.

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Normal Mode Solution to the Equations of Motion of a Flexible Airplane

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Introduction

The equations of motion of a damped linear dynamical system that is idealized by discrete-elements can be expressed by either N second-order Lagrangian equations or by an equivalent set of $2N$ first-order Hamiltonian equations. For small oscillations, the set of N Lagrangian equations can be expressed in the matrix form

$$[m]\{\ddot{q}(t)\} + [c]\{\dot{q}(t)\} + [k]\{q(t)\} = \{F(t)\} \quad (1)$$

where $[m]$, $[c]$, and $[k]$ are $N \times N$ symmetrical inertia, damping and stiffness matrices and where $\{q(t)\}$ and $\{F(t)\}$ are $1 \times N$ column matrices of generalized nodal displacements and equivalent forces, respectively. The equivalent set of $2N$ Hamiltonian equations can be expressed in the partitioned matrix form[1]

$$\begin{bmatrix}[O] & [m]\\ [m] & [c]\end{bmatrix}\begin{bmatrix}\{\ddot{q}(t)\}\\ \{\dot{q}(t)\}\end{bmatrix} + \begin{bmatrix}-[m] & [O]\\ [O] & [k]\end{bmatrix}\begin{bmatrix}\{\dot{q}(t)\}\\ \{q(t)\}\end{bmatrix} = \begin{bmatrix}\{O\}\\ \{F(t)\}\end{bmatrix} \quad (2)$$

where the generalized momenta $\{p\} = [m]\{\dot{q}\}$ are used as a set of auxiliary variables and [0] is an $N \times N$ null matrix.

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For aeronautical applications, the generalized equivalent forces are given by

$$\{F(t)\} = \{A(t)\} + \{Q(t)\} \tag{3}$$

where $\{A(t)\}$ and $\{Q(t)\}$ are $1 \times N$ column matrices of generalized equivalent aerodynamic and disturbing forces, respectively. The disturbing forces are generally nonconservative and can result, e.g., from forces on control surfaces, gust loads, or landing loads. In many applications, the aerodynamic forces are approximated by a linear function of $\{q(t)\}$, $\{\dot{q}(t)\}$ and $\{\ddot{q}(t)\}$ at most. The aerodynamic forces can then be expressed in the matrix form[2,3]

$$\{A(t)\} = [A\ddot{q}]\{\ddot{q}(t)\} + [A\dot{q}]\{\dot{q}(t)\} + [Aq]\{q(t)\} \tag{4}$$

where $[A\ddot{q}]$, $[A\dot{q}]$, and $[Aq]$ are $N \times N$ square matrices of aerodynamic derivatives and functions of dynamic pressure, Mach number and frequency.

Substituting Eq. (4) into (3) and the result into Eq. (1) gives

$$[m - A\ddot{q}]\{\ddot{q}(t)\} + [c - A\dot{q}]\{\dot{q}(t)\} + [k - Aq]\{q(t)\} = \{Q(t)\}$$

which represents the set of N Lagrangian equations for a flexible airplane. This set of equations can be put in the form of a proportional viscous damping problem with symmetrical coefficients and solved by the classical normal mode method only for those applications where it can be assumed that $[A\ddot{q}]$ and $[Aq]$ are symmetrical and $[c - A\dot{q}]$ is proportional to a linear function of $[m - A\ddot{q}]$ and $[k - Aq]$.[4] Substituting Eq. (4) into (3) and the result into Eq. (2) gives

$$[M]\{\dot{y}(t)\} + [K]\{y(t)\} = \{Y(t)\} \tag{5}$$

where

$$[M] = \begin{bmatrix} [O] & [m] \\ [m - A\ddot{q}] & [c - A\dot{q}] \end{bmatrix} \cdot \quad [K] = \begin{bmatrix} -[m] & [O] \\ [O] & [k - Aq] \end{bmatrix}$$

$$\{y(t)\} = \begin{bmatrix} \{\dot{q}(t)\} \\ \{q(t)\} \end{bmatrix} \quad \cdot \quad \{Y(t)\} = \begin{bmatrix} \{O\} \\ \{Q(t)\} \end{bmatrix}$$

which represents the set of $2N$ Hamiltonian equations for flexible airplane. This set of equations can be put in the form of a nonproportional viscous damping problem with symmetrical coefficients and solved by a normal mode method only for those applications where it can be assumed that $[A\dot{q}]$ and $[Aq]$ are symmetrical and $[A\ddot{q}]$ is either null or symmetrical.[5] In the latter case, the quantity $[A\ddot{q}]\{\dot{q}(t)\}$ is added to and subtracted from the first N equations of Eq. (5).

The purpose of the present analysis is to show that the equations of motion can be solved by a normal mode method in the general case where $[A\ddot{q}]$, $[A\dot{q}]$, and $[Aq]$ are $N \times N$ square matrices. This corresponds to a normal mode solution to a nonproportional viscous damping problem with nonsymmetrical matrix coefficients and is an alternative to treating this class of problem by a transform method. The result sought is the first integral of the Hamiltonian equations of motion. From the result, internal forces and stresses due to dynamic loads can be obtained by the usual methods for both deterministic and probabilistic disturbing forces.

Analysis

Equation (5) is a matrix representation of the equations of motion for the dynamical system of interest. For the purposes of this analysis, $[A\ddot{q}]$, $[A\dot{q}]$, and $[Aq]$ are considered $N \times N$ square matrices. Therefore, $[M]$ and $[K]$ are generally $2N \times 2N$ square matrices unless $[A\ddot{q}]$, $[A\dot{q}]$, and $[Aq]$ are null.

The homogeneous solution to Eq. (5) is obtained by setting $\{Y(t)\}$ equal to zero and assuming a solution of the form

$$\{y(t)\} = \{\Phi\}e^{\lambda t}$$

This leads to the eigenvalue problem

$$\lambda[M]\{\Phi\} + [K]\{\Phi\} = \{O\} \tag{6}$$

which has a nontrival solution only if

$$|\lambda[M] + [K]| = O \tag{7}$$

Equation (7) can be solved for the set of $2N$ eigenvalues λ_r, where $r = 1, \ldots, 2N$. Substituting these eigenvalues into Eq. (6) gives

$$\lambda_r[M]\{\Phi^{(r)}\} + [K]\{\Phi^{(r)}\} = \{O\} \tag{8}$$

which can be solved for the corresponding set of eigenvectors $\{\Phi^{(r)}\}$, where $r = 1, \ldots, 2N$. For physical problems, the eigenvalues are usually distinct and the corresponding eigenvectors are, therefore, linearly independent. The eigenvectors are called modal columns and since $\{\dot{y}(t)\} = \lambda\{y(t)\}$, it follows that

$$\begin{bmatrix} \{{}_{(r)}\phi\} \\ \{{}_{(r)}\phi\}^{r}\lambda \end{bmatrix} = \{{}_{(r)}\Phi\}$$

Associated with the eigenvalue problem is the adjoint eigenvalue problem[6]

$$\lambda\{\Psi\}^T[M] + \{\Psi\}^T[K] = \{O\} \tag{9}$$

Since the eigenvalue problem and the adjoint eigenvalue problem have the same characteristic equation, Eq. (7), it follows that they have the same set of eigenvalues. Substituting the eigenvalues into Eq. (9) gives

$$\lambda_s\{\Psi^{(s)}\}^T[M] + \{\Psi^{(s)}\}^T[K] = \{O\} \tag{10}$$

which can be solved for the adjoint eigenvectors $\{\Psi^{(s)}\}$ that correspond to λ_s, where $s = 1, \ldots, 2N$. The transpose of the adjoint eigenvectors $\{\Psi^{(s)}\}^T$ are called modal rows. It is noted that the eigenvectors $\{\Phi^{(r)}\}$ and the adjoint eigenvectors $\{\Psi^{(s)}\}$ are called conjugates for $r = s$ and that the conjugate eigenvectors are identical and called self-adjoint if $[M]$ and $[K]$ are symmetrical. The conjugate eigenvectors are, therefore, self-adjoint if $[A\ddot{q}]$, $[A\dot{q}]$, and $[Aq]$ are null. In this case, Eq. (9) is the transpose of Eq. (6).

Premultiplying Eq. (8) by $\{\Psi^{(s)}\}^T$, postmultiplying Eq. (10) by $\{\Phi^{(r)}\}$, and subtracting the two results gives

$$\{\Psi^{(s)}\}^T[M]\{\Phi^{(r)}\} = \{O\} \tag{11}$$

for $r \neq s$, assuming the eigenvalues are distinct. Premultiplying Eq. (8) by $\{\Psi^{(s)}\}^T$ and applying Eq. (11) gives

$$\{\Psi^{(s)}\}^T[K]\{\Phi^{(r)}\} = \{O\} \tag{12}$$

for $r \neq s$. Equations (11) and (12) are biorthogonality relations and are generalized by letting

$$\{\Psi^{(s)}\}^T[M]\{\Phi^{(r)}\} = \delta_{rs}M_r^* \text{ and } \{\Psi^{(s)}\}^T[K]\{\Phi^{(r)}\} = \delta_{rs}K_r^* \tag{13}$$

where

$$K_r^* = -\lambda_r M_r^* \quad \text{and} \quad \delta_{rs} = \begin{cases} 0, & r \neq s \\ 1, & r = s \end{cases}$$

The nonhomogeneous solution to Eq. (5) is obtained by letting

$$\{y(t)\} = \sum_{r=1}^{2N} \{\Phi^{(r)}\}\eta_r(t) \tag{14}$$

where $\eta_r(t)$ are a set of, yet to be determined, normal coordinates. Substituting Eq. (14) into (5) and premultiplying the result by $\{\Psi^{(s)}\}^T$ gives

$$\sum_{r=1}^{2N} \{\Psi^{(s)}\}^T[M]\{\Phi^{(r)}\}\dot{\eta}_r(t) + \sum_{r=1}^{2N} \{\Psi^{(s)}\}^T[K]\{\Phi^{(r)}\}\eta_r(t) = \{\Psi^{(s)}\}^T\{Y(t)\}$$

which, in view of Eq. (13), reduces to the set of $2N$ uncoupled equations

$$\dot{\eta}_s(t) - \lambda_s \eta_s(t) = \frac{1}{M_s{}^*} F_s{}^*(t) \qquad (15)$$

where

$$F_s{}^*(t) = \{\Psi^{(s)}\}^T\{Y(t)\}$$

for $s = 1, \ldots, 2N$. The particular solution of Eq. (15) is given by the convolution integral

$$\eta_s(t) = \frac{1}{M_s{}^*} \int_0^t e^{\lambda_s \tau} F_s{}^*(t-\tau)d\tau \qquad (16)$$

and thus the set of normal coordinates are determined.

Finally, substituting Eq. (16) into (14) gives

$$\{y(t)\} = \sum_{r=1}^{2N} \frac{\{\Phi^{(r)}\}}{M_r{}^*} \int_0^t e^{\lambda_r \tau} F_r{}^*(t-\tau)d\tau$$

which, in terms of the elements of $\{y(t)\}$ and $\{\Phi^{(r)}\}$, is

$$\begin{bmatrix} \{\dot{q}(t)\} \\ \{q(t)\} \end{bmatrix} = \sum_{r=1}^{2N} \frac{1}{M_r{}^*} \begin{bmatrix} \lambda_r\{\phi^{(r)}\} \\ \{\phi^{(r)}\} \end{bmatrix} \int_0^t e^{\lambda_r \tau} F_r{}^*(t-\tau)d\tau \qquad (17)$$

Equation (17) is the result sought.

Conclusion

It is clear that the steady-state generalized nodal response is simply

$$\{q(t)\} = \sum_{r=1}^{2N} \frac{\{\phi^{(r)}\}}{M_r{}^*} \int_0^t e^{\lambda_r \tau} F_r{}^*(t-\tau)d\tau \qquad (18)$$

For stable systems, the eigenvalues are complex with a negative real part for underdamped modes. In this case, both the eigenvalues and eigenvectors occur in complex conjugate pairs and Eq. (18) can be expressed in real form by N terms.

In conclusion, this paper points out that the equations of motion of a flexible airplane can be written in the form of a general nonself-adjoint system which can be solved by the eigenfunction expansion.

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Errata

Drift of Buoyant Wing-Tip Vortices

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Equations (21, 23, and 24) should be corrected to read

$$\Gamma^2/a^3g > 4\pi^2[1 + (\lambda/2)] \qquad (21)$$

$$\frac{\Gamma^2}{a^3g} > \begin{cases} 4\pi^2[1 + (\lambda/2)] & (\lambda < 0.772) \\ 7.18\pi^2\lambda & (\lambda > 0.772) \end{cases} \qquad (23)$$

$$\langle \dot{X} \rangle_{\max} = \begin{cases} -(\lambda/2)\{ag/[1 + (\lambda/2)]\}^{1/2} & (\lambda < 0.772) \\ -(ag\lambda/7.18)^{1/2} & (\lambda > 0.772) \end{cases} \qquad (24)$$

These corrections do not alter any of the other results or conclusions.

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